

On the chromatic numbers of spheres in \mathbb{R}^{n*}

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1 Introduction

In this paper, we study a classical problem going back to H. Hadwiger, E. Nelson, and P. Erdős. Let (X, ρ) be a metric space. Consider a set \mathcal{A} of distinct positive reals. We call the value

$$\chi((X, \rho); \mathcal{A}) = \min \left\{ \chi : X = X_1 \bigsqcup \dots \bigsqcup X_\chi, \forall i \forall x, y \in X_i \quad \rho(x, y) \notin \mathcal{A} \right\}$$

the chromatic number of the space (X, ρ) with the set of forbidden distances \mathcal{A} . In other words, $\chi((X, \rho); \mathcal{A})$ is the minimum number of colours needed to paint all the points in X so that any two points at a distance from \mathcal{A} apart receive different colours.

Various metric spaces and sets of forbidden distances have been considered by many authors. Let us briefly review the most important cases.

1. $(X, \rho) = (\mathbb{R}^n, l_2)$, $\mathcal{A} = \{1\}$. Here

$$l_2(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2},$$

$$\mathbf{x} = (x_1, \dots, x_n), \quad \mathbf{y} = (y_1, \dots, y_n).$$

This is the classical case, which is deeply investigated. We use a simpler standard notation $\chi(\mathbb{R}^n)$ for the corresponding chromatic number. Numerous results concerning $\chi(\mathbb{R}^n)$ can be found in the books [1], [2] and surveys [3], [4]. For our further

*This work is done under the financial support of the following grants: the grant 09-01-00294 of Russian Foundation for Basic Research, the grant MD-8390.2010.1 of the Russian President, the grant NSh-691.2008.1 supporting Leading scientific schools of Russia, a grant of Dynastia foundation.

purposes, only the following bounds will be useful:

$$\begin{aligned}\chi(\mathbb{R}^n) &\geq (\zeta_1 + o(1))^n, \quad \zeta_1 = \frac{1 + \sqrt{2}}{2} = 1.207\dots \quad (\text{see [5]}), \\ \chi(\mathbb{R}^n) &\geq (\zeta_2 + o(1))^n, \quad \zeta_2 = 1.239\dots \quad (\text{see [6]}), \\ \chi(\mathbb{R}^n) &\leq (3 + o(1))^n \quad (\text{see [7]}).\end{aligned}$$

2. $(X, \rho) = (\mathbb{R}^n, l_2)$, $|\mathcal{A}| = k$, $k \in \mathbb{N}$. Here the best known results are given in the paper [8].
3. $(X, \rho) = (\mathbb{R}^n, l_2)$, $|\mathcal{A}| = \infty$. Here the paper [9] should be cited.
4. $(X, \rho) = (\mathbb{R}^n, l_p)$, $|\mathcal{A}| = k$, $k \in \mathbb{N}$, where

$$l_p(\mathbf{x}, \mathbf{y}) = \sqrt[p]{|x_1 - y_1|^p + \dots + |x_n - y_n|^p}, \quad p \in [1, \infty),$$

$$l_\infty(\mathbf{x}, \mathbf{y}) = \max_{i=1, \dots, n} |x_i - y_i|.$$

These cases were studied in [10], [11], [12], [13].

5. $(X, \rho) = (\mathbb{Q}^n, l_p)$, $|\mathcal{A}| = k$, $k \in \mathbb{N}$. See [4], [13], [14], [15] for multiple references.

Another interesting series of metric spaces is generated by spheres S_r^{n-1} of radii $r \geq \frac{1}{2}$ in \mathbb{R}^n : $(X, \rho) = (S_r^{n-1}, l_2)$, $\mathcal{A} = \{1\}$. Studying

$$\chi(S_r^{n-1}) = \chi((S_r^{n-1}, l_2); \{1\})$$

was proposed by Erdős who conjectured in [16] that $\chi(S_r^{n-1}) \rightarrow \infty$ for any fixed value of $r > \frac{1}{2}$. It is obvious that $\chi(S_{1/2}^{n-1}) = 2$, and L. Lovász proved Erdős' conjecture in [17] using topological tools (see also [18]). The exact assertion of Lovász is as follows: *for any $r > \frac{1}{2}$ and $n \in \mathbb{N}$, the inequality holds $\chi(S_r^{n-1}) \geq n$; if $r < \sqrt{\frac{n}{2n+2}} \sim \frac{1}{\sqrt{2}}$, i.e., the length of any side of a regular n -simplex inscribed into S_r^{n-1} is smaller than 1, then $\chi(S_r^{n-1}) \leq n + 1$.* Although this result is widely cited (see, e.g., [3]), its second part is completely wrong (see Section 5). Actually, for every $r > \frac{1}{2}$, the quantity $\chi(S_r^{n-1})$ grows exponentially, not linearly.

In this paper, we will do a careful analysis of the asymptotic behaviour of the value $\chi(S_r^{n-1})$. We will study even some cases when r may depend on n .

Before proceeding to formulating our main results, let us mention some more papers on the chromatic numbers of spheres: [19], [20].

2 Statements of the main results

The starting point for our investigation is the following assertion.

Theorem 1. *For any $r > \frac{1}{2}$, there exist a constant $\gamma = \gamma(r) > 1$ and a function $\varphi(n) = \varphi(n, r) = o(1)$, $n \rightarrow \infty$, such that for every $n \in \mathbb{N}$, the inequality holds*

$$\chi(S_r^{n-1}) \geq (\gamma + \varphi(n))^n.$$

Theorem 1 sais that, for any fixed radius, the chromatic number grows exponentially in the dimension. Of course it is possible to make the value of $\gamma(r)$ a bit more concrete. The first step in this direction is given in Theorem 2.

Theorem 2. *For any $r \in \left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$, there exists a function $\delta(n) = \delta(n, r) = o(1)$, $n \rightarrow \infty$, such that for every $n \in \mathbb{N}$, the inequality holds*

$$\chi(S_r^{n-1}) \geq \left(2 \left(\frac{1}{8r^2} \right)^{\frac{1}{8r^2}} \left(1 - \frac{1}{8r^2} \right)^{1-\frac{1}{8r^2}} + \delta(n) \right)^n.$$

Looking at Theorem 2, we see that if r becomes closer and closer to $\frac{1}{\sqrt{2}}$, then the constant

$$\gamma = 2 \left(\frac{1}{8r^2} \right)^{\frac{1}{8r^2}} \left(1 - \frac{1}{8r^2} \right)^{1-\frac{1}{8r^2}}$$

approaches the value $\zeta_3 = 1.139\dots$ Since $S_r^{n-1} \subset \mathbb{R}^n$ leading to $\chi(S_r^{n-1}) \leq \chi(\mathbb{R}^n)$, one may not expect that ζ_3 could be somehow replaced by anything greater than ζ_2 (cf. Introduction). However, there is some room to spare here, and in Section 7 we will exhibit a further optimization process providing even larger constants.

At the same time, if $r \geq \frac{1}{\sqrt{2}}$, then we certainly have

$$\chi(S_r^n) \geq \chi(S_{r'}^{n-1}) \geq (1.139 + o(1))^n, \quad r' < \frac{1}{\sqrt{2}} \leq r.$$

So, once again, for *any* fixed value of radius, the chromatic number is essentially exponential in n . Comparing our results with those due to Lovász, we get the following assertion.

Theorem 3. *For any $r > \frac{1}{2}$, there exists an n_0 such that for every $n \geq n_0$, $\chi(S_r^{n-1}) > n + 1$.*

On the one hand, Theorem 3 shows that the bound $\chi(S_r^{n-1}) \leq n + 1$ is false, provided we fix r and let n go to infinity. On the other hand, the result of Theorem 3 is much stronger than that of Lovász only for the values of n which are big enough. So in small dimensions, the *lower* estimate $\chi(S_r^{n-1}) \geq n + 1$ is still the best known (and true).

The gap between exponents and linear functions is quite large. Thus, one may expect that superlinear lower bounds for $\chi(S_r^{n-1})$ would be possible not only for a constant $r > \frac{1}{2}$, but also for some sequences $r_n \rightarrow \frac{1}{2}$. The most general assertion of this kind is in Theorem 4.

Theorem 4. *Let \mathbb{P} be the set of prime numbers. Let $f(x)$ be such a function that for any $x \in \mathbb{R}$, $x \geq 0$,*

$$x + f(x) = \min\{p \in \mathbb{P} : p > x\}.$$

Let

$$m(x) = \max\{m < x : m \equiv 0 \pmod{4}\}.$$

Consider a sequence $\{r_n\}_{n=1}^\infty$, where $r_n > \frac{1}{2}$ for each $n \in \mathbb{N}$. Set

$$p(n) = \frac{m(n)}{8r_n^2} + f\left(\frac{m(n)}{8r_n^2}\right).$$

If

$$\frac{m(n)}{4} < p(n) \leq \frac{m(n)}{2}, \quad n \in \mathbb{N},$$

then,

$$\chi(S_{r_n}^{n-1}) \geq \frac{C_{m(n)}^{m(n)/2}}{C_{m(n)}^{p(n)}}.$$

Translating Theorem 4 into a form of Theorem 3, we get

Theorem 5. Consider a sequence $\{r_n\}_{n=1}^{\infty}$, where $r_n > \frac{1}{2}$ for each $n \in \mathbb{N}$. Let $p(n)$ be the same as in Theorem 4. If

$$\frac{m(n)}{4} < p(n) < \frac{m(n)}{2} - \sqrt{\frac{m(n) \ln m(n)}{\kappa}}, \quad \kappa < 2, \quad n \in \mathbb{N},$$

then,

$$\chi(S_{r_n}^{n-1}) > n + 1, \quad \forall n \geq n_0.$$

The quality of Theorem 5 depends on the estimates for the function $f(x)$. Determining the exact asymptotic behaviour of $f(x)$ is a very hard problem of analytical number theory (see [21]). As far as we know, the best upper estimate is $f(x) = O(x^{0.525-\varepsilon})$ with a so small $\varepsilon > 0$ that the authors did not care of it (see [22]). However, it is conjectured that $f(x) = O(\ln x)$ (see [23]). The tightest lower bound is given in [24] and [25], but it is sublogarithmic and apparently far enough from the truth. Using this information, we may derive

Theorem 6. Assume that $c_0 > 0$ is such that $f(x) \leq c_0 x^{0.525}$ for every x . Then, there exists a constant $c'_0 > 0$ such that for any sequence of radii r_n satisfying the inequality

$$r_n \geq \frac{1}{2} + \frac{c'_0}{n^{0.475}},$$

we have the bound

$$\chi(S_{r_n}^{n-1}) > n + 1, \quad \forall n \geq n_0.$$

Theorem 7. Assume that $c_1 > 0$ is such that $f(x) \leq c_1 \ln x$ for every x . Then, there exists a constant $c'_1 > 0$ such that for any sequence of radii r_n satisfying the inequality

$$r_n \geq \frac{1}{2} + c'_1 \sqrt{\frac{\ln n}{n}},$$

we have the bound

$$\chi(S_{r_n}^{n-1}) > n + 1, \quad \forall n \geq n_0.$$

So $r_n > \frac{1}{2}$ may be quite close to the value $\frac{1}{2}$, and, nevertheless, the chromatic numbers will exceed the Lovász upper estimate. Finally, it is of interest for which sequences of r_n , we do really have the bound $\chi(S_{r_n}^{n-1}) \leq n + 1$.

Theorem 8. *There exists a constant $c_2 > 0$ such that for any sequence of radii r_n satisfying the inequality*

$$r_n \leq \frac{1}{2} + \frac{c_2}{n},$$

we have the bound

$$\chi(S_{r_n}^{n-1}) \leq n + 1, \quad \forall n \geq n_0.$$

Further structure of the paper is as follows. In Section 3, we shall give proofs for Theorems 1 – 4. Section 4 will be devoted to proving Theorems 5 – 7. In Section 5, we shall discuss Theorem 8. In Section 6, some more comments and suggestions will be given. In particular, we shall exhibit more general *upper* estimates for $\chi(S_{r_n}^{n-1})$ than those in Theorem 8. In Section 7, we shall present a general scheme for obtaining better (and, in some sense, optimal) constants γ than those appearing in Theorems 1 and 2.

3 Proofs of Theorems 1 – 4

Among Theorems 1 – 3, Theorem 2 covers both Theorem 1 and Theorem 3. So we start by proving Theorem 2.

Fix an $r \in \left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$ and an $n \in \mathbb{N}$. Let $m < n$ be the maximum natural number which is divisible by 4. Let us find a' from the relation

$$\frac{\sqrt{m}}{\sqrt{2m - 2a'}} = r, \quad \text{i.e.,} \quad a' = \frac{m(2r^2 - 1)}{2r^2}.$$

Let p be the smallest prime number satisfying the inequality

$$p > \frac{m - a'}{4} = \frac{m}{8r^2}.$$

Set

$$a = m - 4p < a'.$$

Consider the following graph $G = (V, E)$:

$$V = \left\{ \mathbf{x} = (x_1, \dots, x_m) : x_i \in \left\{ -\frac{1}{\sqrt{2m-2a}}, \frac{1}{\sqrt{2m-2a}} \right\}, x_1 + \dots + x_m = 0 \right\},$$

$$E = \{ \{\mathbf{x}, \mathbf{y}\} : \mathbf{x}, \mathbf{y} \in V, l_2(\mathbf{x}, \mathbf{y}) = 1 \}.$$

Obviously $V \subset S_r^{m-1}$, where

$$r' = \frac{\sqrt{m}}{\sqrt{2m-2a}} < \frac{\sqrt{m}}{\sqrt{2m-2a'}} = r.$$

If we use the standard notation $\chi(G)$ for the chromatic number of G and $\alpha(G)$ for its independence number, then we get

$$\chi(S_r^{n-1}) \geq \chi(S_{r'}^{m-1}) \geq \chi(G) \geq \frac{|V|}{\alpha(G)} = \frac{C_m^{m/2}}{\alpha(G)}.$$

So we are led to estimate $\alpha(G)$ from above. It is convenient to transform $G = (V, E)$ into an $H = (W, F)$:

$$W = \{ \mathbf{x} \cdot \sqrt{2m-2a} : \mathbf{x} \in V \}, \quad F = \{ \{\mathbf{x}, \mathbf{y}\} : \mathbf{x}, \mathbf{y} \in W, l_2(\mathbf{x}, \mathbf{y}) = \sqrt{2m-2a} \}.$$

Let us denote by (\mathbf{x}, \mathbf{y}) the Euclidean scalar product of \mathbf{x} and \mathbf{y} . Since for any $\mathbf{x} \in W$, $(\mathbf{x}, \mathbf{x}) = m$, we may rewrite F as follows:

$$F = \{ \{\mathbf{x}, \mathbf{y}\} : \mathbf{x}, \mathbf{y} \in W, (\mathbf{x}, \mathbf{y}) = a \}.$$

Notice that for $\mathbf{x}, \mathbf{y} \in W$, the quantity (\mathbf{x}, \mathbf{y}) lies in the interval $[-m, m]$ and is congruent to zero modulo 4. The last observation is due to the fact that $m \equiv 0 \pmod{4}$ and every vector $\mathbf{x} \in W$ contains an even number of negative coordinates. Also,

$$m - 8p < m - \frac{m}{r^2} < -m.$$

Thus, for every $\mathbf{x}, \mathbf{y} \in W$,

$$(\mathbf{x}, \mathbf{y}) \equiv m \pmod{p} \iff (\mathbf{x}, \mathbf{y}) = m \text{ or } (\mathbf{x}, \mathbf{y}) = a. \quad (1)$$

Now, we are about to prove that $\alpha(G) = \alpha(H) \leq C_m^p$. Take an arbitrary

$$Q = \{\mathbf{x}_1, \dots, \mathbf{x}_s\} \subset W, \quad \forall i \forall j, \quad (\mathbf{x}_i, \mathbf{x}_j) \neq a. \quad (2)$$

In other words, Q is an independent set in H . We have to show that $s \leq C_m^p$. For this purpose, we use the linear algebra method (see [5], [26], [27], [28]).

To each vector $\mathbf{x} \in W$ we assign a polynomial $P_{\mathbf{x}} \in \mathbb{Z}/p\mathbb{Z}[y_1, \dots, y_m]$. First, we take

$$P'_{\mathbf{x}}(\mathbf{y}) = \prod_{i \in I} (i - (\mathbf{x}, \mathbf{y})),$$

where

$$I = \{0, 1, \dots, p-1\} \setminus \{m \pmod{p}\}, \quad \mathbf{y} = (y_1, \dots, y_m),$$

and so $P'_{\mathbf{x}} \in \mathbb{Z}/p\mathbb{Z}[y_1, \dots, y_m]$. Obviously,

$$\forall \mathbf{x}, \mathbf{y} \in W \quad P'_{\mathbf{x}}(\mathbf{y}) \equiv 0 \pmod{p} \iff (\mathbf{x}, \mathbf{y}) \not\equiv m \pmod{p}. \quad (3)$$

Second, we represent $P'_{\mathbf{x}}$ as a sum of monomials. If a monomial has the form

$$y_{i_1}^{\alpha_{i_1}} \cdot \dots \cdot y_{i_q}^{\alpha_{i_q}}, \quad \alpha_{i_1} > 0, \dots, \alpha_{i_q} > 0,$$

then we replace it by

$$y_{i_1}^{\beta_{i_1}} \cdot \dots \cdot y_{i_q}^{\beta_{i_q}},$$

where $\beta_\nu = 1$, provided α_ν is odd, and $\beta_\nu = 0$, provided α_ν is even. Eventually, we get a polynomial $P_{\mathbf{x}}$. It is worth noting that this polynomial does also satisfy property (3).

It follows from properties (1), (2), and (3) that the polynomials

$$P_{\mathbf{x}_1}, \dots, P_{\mathbf{x}_s}$$

assigned to the vectors of the set Q are linearly independent over $\mathbb{Z}/p\mathbb{Z}$. It is also easy to see that the dimension of the space generated by

$$P_{\mathbf{x}_1}, \dots, P_{\mathbf{x}_s}$$

does not exceed C_m^p . Thus, $s = |Q| \leq C_m^p$ and, therefore,

$$\chi(S_r^{n-1}) \geq \frac{C_m^{m/2}}{C_m^p}.$$

Standard analytical tools (like Stirling's formula) together with $p \sim \frac{m}{8r^2}$ give us, finally, the expected bound

$$\chi(S_r^{n-1}) \geq \left(2 \left(\frac{1}{8r^2} \right)^{\frac{1}{8r^2}} \left(1 - \frac{1}{8r^2} \right)^{1-\frac{1}{8r^2}} + \delta(n) \right)^n,$$

which completes the proof of Theorems 1 – 3.

The proof of Theorem 4 is now clear. We just reproduce the above argument with r_n instead of r . The only thing one has to explain here is why we impose additional conditions on the value of a prime. Indeed, the inequality $p(n) > \frac{m(n)}{4}$ is quite important, since property (1) becomes false without it. As for the inequality $p(n) \leq \frac{m(n)}{2}$, it is necessary to correctly estimate the independence number of our graph G by the quantity C_m^p . Moreover, $\chi(G) = 1$, provided $p(n) > \frac{m(n)}{2}$, and the result is trivial. Theorem 4 is proved.

4 Proofs of Theorems 5 – 7

4.1 Proof of Theorem 5

Set $m = m(n)$, $p = p(n)$. Since the function $\frac{C_m^{m/2}}{C_m^p}$ is decreasing in p , we just have to show that for

$$p = \left[\frac{m}{2} - \sqrt{\frac{m \ln m}{\kappa}} \right],$$

the inequality $\frac{C_m^{m/2}}{C_m^p} > n + 1$ is true for large values of n . We have

$$\begin{aligned} \frac{C_m^{m/2}}{C_m^p} &= \frac{\left(\frac{m}{2} + 1\right) \cdot \left(\frac{m}{2} + 2\right) \cdot \dots \cdot \left(\frac{m}{2} + \left(\frac{m}{2} - p\right)\right)}{\frac{m}{2} \cdot \left(\frac{m}{2} - 1\right) \cdot \dots \cdot \left(\frac{m}{2} - \left(\frac{m}{2} - p - 1\right)\right)} = \\ &= \frac{\left(1 + \frac{2}{m}\right) \cdot \left(1 + \frac{4}{m}\right) \cdot \dots \cdot \left(1 + \frac{m-2p}{m}\right)}{\left(1 - \frac{2}{m}\right) \cdot \left(1 - \frac{4}{m}\right) \cdot \dots \cdot \left(1 - \frac{m-2p-2}{m}\right)} \sim e^{\frac{(m-2p)^2}{2m}} \geq e^{\frac{2 \ln m}{\kappa}} = m^{\frac{2}{\kappa}}. \end{aligned}$$

By a condition of Theorem 5, $\kappa < 2$. Thus,

$$m^{\frac{2}{\kappa}}(1 + o(1)) > n + 1, \quad \forall n \geq n_0.$$

Theorem 5 is proved.

4.2 Proof of Theorem 6

We just have to show that for our choice of r_n ,

$$p = \frac{m}{8r_n^2} + f\left(\frac{m}{8r_n^2}\right) < \frac{m}{2} - \sqrt{\frac{m \ln m}{\kappa}},$$

provided $\kappa < 2$ is a constant and n is large enough.

Indeed, assume that c'_0 is large (say, $c'_0 > c_0$). Then,

$$\begin{aligned} p &\leq \frac{m}{8\left(\frac{1}{2} + \frac{c'_0}{n^{0.475}}\right)^2} + c_0 \left(\frac{m}{8\left(\frac{1}{2} + \frac{c'_0}{n^{0.475}}\right)^2} \right)^{0.525} < \\ &< \frac{m}{8\left(\frac{1}{4} + \frac{c'_0}{n^{0.475}}\right)} + c_0 \left(\frac{m}{8\left(\frac{1}{4} + \frac{c'_0}{n^{0.475}}\right)} \right)^{0.525} = \\ &= \frac{m}{2} \left(1 - \frac{4c'_0}{n^{0.475}} + o\left(\frac{1}{\sqrt{n}}\right) \right) + c_0 \left(\frac{m}{2} \left(1 - \frac{4c'_0}{n^{0.475}} + o\left(\frac{1}{\sqrt{n}}\right) \right) \right)^{0.525}. \end{aligned}$$

For any sufficiently large value of n , the last quantity is bounded from above by

$$\frac{m}{2} - c'_0 m^{0.525} + c_0 m^{0.525} = \frac{m}{2} - c''_0 m^{0.525}, \quad c''_0 > 0.$$

Obviously, for any $n \geq n_0$,

$$\frac{m}{2} - c''_0 m^{0.525} < \frac{m}{2} - \sqrt{\frac{m \ln m}{\kappa}}.$$

Theorem 6 is proved.

4.3 Proof of Theorem 7

Let us briefly write down a series of inequalities similar to those in 4.2:

$$\begin{aligned} p &\leq \frac{m}{8\left(\frac{1}{2} + c'_1 \sqrt{\frac{\ln n}{n}}\right)^2} + c_1 \ln \left(\frac{m}{8\left(\frac{1}{2} + c'_1 \sqrt{\frac{\ln n}{n}}\right)^2} \right) < \\ &< \frac{m}{2} \left(1 - 4c'_1 \sqrt{\frac{\ln n}{n}} + o\left(\frac{1}{n^{3/2}}\right) \right) + c_1 \ln \left(\frac{m}{2} \left(1 - 4c'_1 \sqrt{\frac{\ln n}{n}} + o\left(\frac{1}{n^{3/2}}\right) \right) \right) < \\ &< \frac{m}{2} - c'_1 \sqrt{m \ln m}, \end{aligned}$$

and we are done.

5 Proof of Theorem 8

Let us take $S_{1/2}^{n-1}$ and divide it into $n+1$ parts of smallest possible diameters. To this end, we inscribe a regular n -simplex Δ^n into $S_{1/2}^{n-1}$ and consider multidimensional polyhedral cones C_1, \dots, C_{n+1} with common vertex at the center of $S_{1/2}^{n-1}$ and coming through the $(n-1)$ -faces of Δ^n . Obviously,

$$S_{1/2}^{n-1} = (S_{1/2}^{n-1} \cap C_1) \cup \dots \cup (S_{1/2}^{n-1} \cap C_{n+1}). \quad (4)$$

In principle, it is a good exercise in multidimensional geometry to prove that for any i ,

$$\text{diam } (S_{1/2}^{n-1} \cap C_i) = 1 - \Theta\left(\frac{1}{n}\right).$$

It follows immediately from this observation that we may inflate $S_{1/2}^{n-1}$ at most

$$\frac{1}{1 - \Theta\left(\frac{1}{n}\right)} = 1 + \Theta\left(\frac{1}{n}\right)$$

times in order to get a partition of the resulting sphere into parts of diameter not exceeding 1. Thus, for a constant $c_2 > 0$, we have an appropriate coloring of $S_{1/2+c_2/n}^{n-1}$, which completes the proof of Theorem 8.

Apparently, in [17], the same construction was proposed. However, the author assumed that the diameter of any part in the corresponding partition is attained on the sides of a regular n -simplex Δ^n . This is true only for $n = 2$. Already in \mathbb{R}^3 , the diameter of a part is $\sqrt{\frac{3+\sqrt{3}}{6}} = 0.888\dots$, which is not the length of a side of a tetrahedron inscribed into $S_{1/2}^2$.

6 Comments and upper bounds

First of all, it is worth noting that there is still a certain gap between the estimates

$$r_n \geq \frac{1}{2} + c'_1 \sqrt{\frac{\ln n}{n}} \quad (5)$$

and

$$r_n \leq \frac{1}{2} + \frac{c_2}{n}. \quad (6)$$

Removing this gap could be a good problem. As for (6), it cannot be enlarged by any refinement of the techniques of the previous section. The point is that the partition (4) is best possible: for any other decomposition of $S_{1/2}^{n-1}$ into $n+1$ parts, there exists a part whose diameter is not less than each of the diameters $\text{diam } (S_{1/2}^{n-1} \cap C_i)$. Of course it is not necessary to divide a sphere into parts with diameters strictly smaller than 1; we just need to cut it in such a way that no part would contain a pair of points at the unit distance. However, we do not know such a partition. Perhaps it is easier to improve (5). One should combine linear algebra of Section 3 with some additional ideas.

Let us say a few words about general upper estimates for $\chi(S_{r_n}^{n-1})$. The simplest observation here is that

$$\chi(S_{r_n}^{n-1}) \leq \chi(\mathbb{R}^n) \leq (3 + o(1))^n \quad (\text{cf. Introduction}). \quad (7)$$

Thus, for constant values $r > \frac{1}{2}$ (as in Theorems 1 – 3), we already get the order of magnitude for any quantity $\log \chi(S_r^{n-1})$.

In [29], C.A. Rogers proved that any sphere of radius r in \mathbb{R}^n can be covered by $\left(\frac{r}{\rho} + o(1)\right)^n$ spheres of radius $\rho < r$. In our case, this means that

$$\chi(S_{r_n}^{n-1}) \leq (2r_n + o(1))^n.$$

If $r_n < 3/2$, then this bound is better than that in (7).

More precisely, Rogers' estimate is as follows: *there is an absolute constant $c > 0$ such that, if $r > \frac{1}{2}$ and $n \geq 9$, any n -dimensional spheres of radius r can be covered by less than $cn^{5/2}(2r)^n$ spheres of radius $\frac{1}{2}$.* A so precise formulation is unuseful when r is a constant, but coming again to $r_n \rightarrow \frac{1}{2}$ we may carefully apply this statement in order to obtain upper bounds like

$$\chi(S_{r_n}^{n-1}) \leq 2cn^{5/2}(2r_n)^n = \Theta\left(n^{5/2}(2r_n)^n\right). \quad (8)$$

Here the factor 2 is due to the fact that $\chi(S_{1/2}^{n-1}) = 2$. One should not forget that if, for example, $r_n = \frac{1}{2} + \Theta\left(\frac{1}{n}\right)$, then $(2r_n)^n = \Theta(1)$, so that estimate (8) is very good.

It is possible to evaluate even more sophisticated bounds for $\chi(S_{r_n}^{n-1})$, but this is not so interesting.

7 A possible way for improving Theorem 2

7.1 Statements of the results

Fix again an $r > \frac{1}{2}$. Let $m = m(n) < n$ for every n and $m \sim n$ for $n \rightarrow \infty$. Assume that $t = t(n) \in \mathbb{N}$,

$$b_1 = b_1(n) \in \mathbb{Z}, \dots, b_t = b_t(n) \in \mathbb{Z},$$

$$l_1 = l_1(n) \in \mathbb{N}, \dots, l_t = l_t(n) \in \mathbb{N}, l_1 + \dots + l_t = m.$$

Consider

$$V = V(n) = \{\mathbf{x} = (x_1, \dots, x_m) : x_i \in \{b_1, \dots, b_t\}, |\{i : x_i = b_j\}| = l_j, j = 1, \dots, t\}.$$

Let $d = d(n)$ be the maximum natural number such that for any $\mathbf{x}, \mathbf{y} \in V$, we have $(\mathbf{x}, \mathbf{y}) \equiv 0 \pmod{d}$. Note that V is an obvious analog of the set W from Section 3, where d was equal to 4. Set

$$\bar{s} = \bar{s}(n) = \max_{\mathbf{x}, \mathbf{y} \in V} (\mathbf{x}, \mathbf{y}), \underline{s} = \underline{s}(n) = \min_{\mathbf{x}, \mathbf{y} \in V} (\mathbf{x}, \mathbf{y}).$$

Find $a' = a'(n)$ from the relation

$$\frac{\sqrt{\bar{s}}}{\sqrt{2\bar{s} - 2a'}} = r.$$

Define $p = p(n)$ as the minimum prime number satisfying the inequality

$$p > \frac{\bar{s} - a'}{d}.$$

Finally, we choose $a = a(n)$ from the condition

$$p = \frac{\bar{s} - a}{d}, \text{ i.e., } a = \bar{s} - dp < a'.$$

We get the following theorem.

Theorem 9. *If $a > \underline{s}$ and $\bar{s} - 2dp < \underline{s}$, then*

$$\chi(S_r^{n-1}) \geq \frac{L}{M},$$

where

$$L = \frac{m!}{l_1! \cdot \dots \cdot l_t!}, \quad M = \sum_{(s_1, \dots, s_t) \in \mathcal{A}} \frac{m!}{s_1! \cdot \dots \cdot s_t!},$$

$$\mathcal{A} = \{(s_1, \dots, s_t) : s_i \in \mathbb{N} \cup \{0\}, s_1 + \dots + s_t = m, s_1 + 2s_2 + \dots + (t-1)s_{t-1} \leq p-1\}.$$

In Theorem 9 we optimize over the parameters t , b_1, \dots, b_t , and l_1, \dots, l_t . This optimization can be a bit simpler, provided we suppose that $l_i \sim l_i^0 n$, where $l_i^0 \in (0, 1)$. Actually this does not substantially change results. In our case, we get

Corollary. *The estimate holds*

$$\chi(S_r^{n-1}) \geq \left(\frac{L_0}{M_0} + o(1) \right)^n,$$

where

$$L_0 = e^{-l_1^0 \ln l_1^0 - \dots - l_t^0 \ln l_t^0}, \quad M_0 = \max_{(s_1^0, \dots, s_t^0) \in \mathcal{A}_0} e^{-s_1^0 \ln s_1^0 - \dots - s_t^0 \ln s_t^0},$$

$$\mathcal{A}_0 = \left\{ (s_1^0, \dots, s_t^0) : s_i^0 \in (0, 1), s_1^0 + \dots + s_t^0 = 1, s_1^0 + 2s_2^0 + \dots + (t-1)s_{t-1}^0 \leq \frac{p}{n} \right\}.$$

We shall prove Theorem 9 in §7.2. Corollary can be easily derived from Theorem 9 using Stirling's formula and other standard tools of analysis.

In this paper, we shall not evaluate optimization from Corollary. Here we only cite the papers [8], [13], in which similar optimization procedures were carefully realized.

7.2 Proof of Theorem 9

Let us start by noting that all the parameters in Theorem 9 are chosen to generalize the approach that we used in Section 3. We have already mentioned that the quantity d plays the role of the number 4 in the corresponding argument. Almost all the other notations are also completely parallel to those appearing in Section 3. Here only m should be replaced by \bar{s} , and we just consider V as an analog to W , without introducing two similar sets V and W as it was done in Section 3.

Set $G = (V, E)$ with

$$E = \{\{\mathbf{x}, \mathbf{y}\} : \mathbf{x}, \mathbf{y} \in V, (\mathbf{x}, \mathbf{y}) = a\}.$$

We think it is now obvious that

$$\chi(S_r^{n-1}) \geq \chi(G) \geq \frac{|V|}{\alpha(G)} = \frac{L}{\alpha(G)}.$$

So it remains to prove that $\alpha(G) \leq M$. This is done by the same linear algebra method as in Section 3.

Indeed, by the conditions of Theorem 9, we have, for every $\mathbf{x}, \mathbf{y} \in V$,

$$(\mathbf{x}, \mathbf{y}) \equiv \bar{s} \pmod{p} \iff (\mathbf{x}, \mathbf{y}) = \bar{s} \text{ or } (\mathbf{x}, \mathbf{y}) = a. \quad (1')$$

Take an arbitrary

$$Q = \{\mathbf{x}_1, \dots, \mathbf{x}_s\} \subset V, \quad \forall i \forall j, \quad (\mathbf{x}_i, \mathbf{x}_j) \neq a. \quad (2')$$

We are about to show that $s \leq M$.

To each vector $\mathbf{x} \in V$ we assign a polynomial $P_{\mathbf{x}} \in \mathbb{Z}/p\mathbb{Z}[y_1, \dots, y_m]$. First, we take

$$P'_{\mathbf{x}}(\mathbf{y}) = \prod_{i \in I} (i - (\mathbf{x}, \mathbf{y})),$$

where

$$I = \{0, 1, \dots, p-1\} \setminus \{\bar{s} \pmod{p}\}, \quad \mathbf{y} = (y_1, \dots, y_m),$$

and so $P'_{\mathbf{x}} \in \mathbb{Z}/p\mathbb{Z}[y_1, \dots, y_m]$. Obviously,

$$\forall \mathbf{x}, \mathbf{y} \in W \quad P'_{\mathbf{x}}(\mathbf{y}) \equiv 0 \pmod{p} \iff (\mathbf{x}, \mathbf{y}) \not\equiv \bar{s} \pmod{p}. \quad (3')$$

Second, we represent $P'_{\mathbf{x}}$ as a sum of monomials. We use the fact that

$$(y_i - b_1) \cdot (y_i - b_2) \cdot \dots \cdot (y_i - b_t) = 0,$$

for any $\mathbf{y} \in V$. So we get a polynomial $P_{\mathbf{x}}$ of degree $< t$. It is worth noting that this polynomial does also satisfy property (3').

It follows from properties (1'), (2'), and (3') that the polynomials

$$P_{\mathbf{x}_1}, \dots, P_{\mathbf{x}_s}$$

assigned to the vectors of the set Q are linearly independent over $\mathbb{Z}/p\mathbb{Z}$. Now it is easy to see that the dimension of the space generated by

$$P_{\mathbf{x}_1}, \dots, P_{\mathbf{x}_s}$$

does not exceed M . Thus, $s = |Q| \leq M$ and, therefore, Theorem 9 is proved.

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